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# ON THE USE OF MFS IN LINEAR INVERSE DIFFUSION PROBLEMS

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***Abstract.*** *This work deals with the estimation of the spatially source term distribution in a multidimensional linear diffusion problem. This work can be physically associated with the detection of material non-homogeneities such as inclusions, obstacles or cracks, heat conduction, groundwater flow detection, and tomography. The solution technique applied in this paper to the inverse problem under consideration relies on the use of the Method of Fundamental Solutions. Non-intrusive measurements were employed and the accuracy of the solution approach was examined by using simulated measurements containing random errors in the inverse analysis. The uniqueness of the problem is also investigated.*

***Keywords***. *Meshless Methods, MFS, Linear Heat Conduction, Inverse Problems*

## 1. Introduction

The problem of sources reconstruction from boundary measurements for the Laplace operator, and more generally in elliptic problems, has been studied by El Badia and Ha Duong (1998). They demonstrate that in general sources are unattainable except for its harmonic components. So the knowledge of the Dirichlet-Newmann map, which in the experimental framework means to formulate different boundary value problems with the same source and overdetermine in the least-squares sense the reconstruction problem, does not increase the existing information about the source. In mathematical terms we can say that for the same Dirichlet data two sources with different normal components with respect to the set of harmonic square integrated functions will create the same Newmann data in the boundary. With this limitation in the reconstruction problem, the problem is whether it is possible to determine the source if some *a priori* knowledge is assumed, that is some sourcelet information. We can distinguish the case in which the source is a product of a known function and an unknown one, such as the base and the height. Due to the nature of the space where the boundary visible component of the source lives, that is, the Harmonic functions space, the Method of Fundamental Solution creates a natural framework for this reconstruction problem.

## 2. MFS Background

The MFS is a straightforward method that allows approaching the solution of boundary value problems for a partial differential equation whenever its fundamental solution is known. The first contributions to this field can be dated back to the 1960's with Kupradze and Alekside (1964) – for an overview a good reference still is Fairweather and Karageorghis (1998). However, due to ill conditioning problems and to the development of other methods, like the Finite Element Method, the MFS was not extensively applied and its rigorous mathematical justification was not proceeded. Over the last decade, the main advantage of the MFS and other meshless methods, namely, its simple computational implementation has been recognized. This work applies the features of the MFS to the treatment of inverse source problems.

In this paper, we considered a linear heat conduction with constant thermophysical properties over a finite medium $\Omega$, subjected to prescribed temperature at the boundary $\partial\Omega$. For the sake of simplicity, let us consider the following geometry

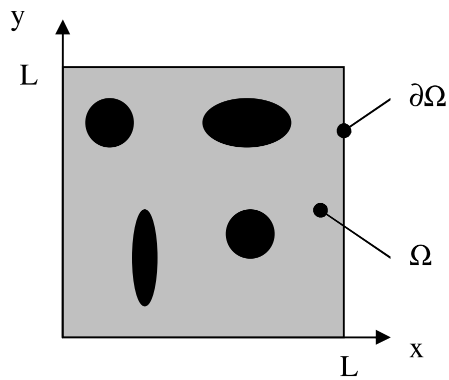



Figure 1. Geometry

which represents a square plate with side equals to *L*. The black patterns represent the location of unknown source terms and all the walls are subjected to Dirichlet boundary conditions. Thus, the mathematical formulation for this problem is given as

$$\nabla \cdot \left( K \nabla T \right) = g \qquad \text{in } \Omega \tag{1.a}$$

$$T = T_0 \qquad \text{at } \partial\Omega \tag{1.b}$$

The problem given by Eqs. (1.a) and (1.b) exists and have unique solution if the geometry, the boundary condition, the thermal conductivity *K* and the source term *g* are known. If, at least, one of these quantities is unknown, we must provide extra information in order to solve this problem. In this case, we don't know the source term *g*, and such extra information can be obtained by the measurements of the heat flux at the boundary, given as the following Neumann boundary condition

$$-K \frac{\partial T}{\partial \mathbf{n}} = q_{meas} \qquad \text{at } \partial\Omega \tag{1.c}$$

where $q_{meas}$ is the measured heat flux.

In a previous paper (Colaço et al, 2003), *g* was estimated by using intrusive measurements in a one-dimensional transient problem, through the Conjugate Gradient Method with Adjoint Problem Technique (Alifanov, 1994; Ozisik and Orlande, 2000). For that technique, the estimation of the source term by using only measurements at the boundary was unsatisfactory. In this paper we are interested in obtain *g* by using only non-intrusive measurements.

Considering a constant thermal conductivity *K* the problem given by Eq. (1.a) reduces to an inverse Poisson problem, that consists in the identification of $f = g/K$ .

$$\nabla \cdot \nabla T = f \qquad \text{in } \Omega \tag{2.a}$$

$$T = T_0 \qquad \text{at } \partial\Omega \tag{2.b}$$

To this purpose we will use solutions of the Helmholtz equations

$$\nabla^2 u + \kappa^2 u = 0$$

since for some chosen wavenumbers $\kappa$ the solutions of these equations simplify the Laplacian to $\nabla \phi^2 u = -\kappa^2 u$ .

One class of solutions is given by point-sources $\phi_k\ (x-y_j)$ with $y_j$ outside the domain. Here $\phi_k$ is the fundamental solution

$$\phi_\kappa(x) = \frac{i}{4} H_0^{(1)}\left[\kappa \|x\|\right] \quad \text{in 2D} \tag{3}$$

where $H_0=J_0+iY_0$ is the first Hänkel function, defined through the Bessel functions of first and second kind, $J_0$ and $Y_0$. Thus, if we consider $T$ as an expansion of point-sources (cf. Alves and Chen (2005))

$$T(x) = \sum_{i=1}^{M}\sum_{j=1}^{N} \beta_{i,j}\phi_{\kappa_i}\left(x-y_j\right) \tag{4}$$

we obtain

$$f(x) = \nabla^2 T = -\sum_{i=1}^{M}\sum_{j=1}^{N} \kappa_i^2 \beta_{i,j}\phi_\kappa\left(x-y_j\right) \tag{5}$$

As adopted by Alves and Valtchev (2005), instead of the whole fundamental solution with external points we may use the imaginary part that reduces to the $J_0$ function, that is non-singular, and allows us to consider internal points inside $\Omega$ (this is also related to the Boundary Knot Method, used by Chen (2001)). The use of these approaches produces similar results.

We just have to adapt $T$ to the given/measured boundary conditions (Cauchy data). Taking the normal derivative of Eq. (4) we obtain

$$\frac{\partial T(x,y)}{\partial \mathbf{n}} = -\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\mathbf{n}\cdot\mathbf{r}_j}{\|\mathbf{r}_j\|}\kappa_i\beta_{i,j}J_1\left(\kappa_i\|\mathbf{r}_j\|\right) \tag{6}$$

with $\|\mathbf{r}_j\|=\|x-y_j\|$. Thus, applying Eqs. (4) and (6) the boundary conditions (1.b) and (1.c) we have

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \beta_{i,j}J_0\left(\kappa_i\|\mathbf{r}_j\|\right) = T_0 \qquad \text{at } \partial\Omega \tag{7}$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\mathbf{n}\cdot\mathbf{r}_j}{\|\mathbf{r}_j\|}\kappa_i\beta_{i,j}J_1\left(\kappa_i\|\mathbf{r}_j\|\right) = \frac{q_{meas}(x,y)}{K} \qquad \text{at } \partial\Omega \tag{8}$$

Note that we must write one pair of equations for each collocation point $(x,y)$, which is the point at the boundary $\partial\Omega$ where we impose the temperature $T_0$ and measure the heat flux $q_{meas}$. Thus, we have $2P$ equations, where $P$ is the number of collocation points $(x,y)$. Since we know the location of the centers $(x_i,y_i)$, the imposed temperature $T_0$, the measured heat flux $q_{meas}$, the thermal conductivity $K$ and have fixed the number of frequencies $\kappa_i$, Eqs. (7) and (8) reduce to a linear system with $M$x$N$ unknowns $\beta_{i,j}$ and $2P$ equations. Or, in matrix form

$$\mathbf{J}\boldsymbol{\beta} = \boldsymbol{\varphi} \tag{9}$$

Since we have a non square matrix, we must solve a least squares problem given as

$$\mathbf{J}^T\mathbf{J}\boldsymbol{\beta} = \mathbf{J}^T\boldsymbol{\varphi} \tag{10}$$

Note that, after solving such system we obtain an expression for the temperature, which was given by Eq. (4). Then, we can take the Laplacian of this expression and obtain

$$-\sum_{i=1}^{M}\sum_{j=1}^{N} \kappa_i^2\beta_{i,j}J_0\left(\kappa_i\|\mathbf{r}_j\|\right) = \nabla^2 T \tag{11}$$

where we used Eq. (5) to obtain this compact form.

Since the conductivity $K$ is constant, we may apply Eq. (11) to Eq. (1.a) and obtain, finally the unknown source term as

$$g\left(x,y\right)=-K\sum_{i=1}^{M}\sum_{j=1}^{N}\kappa_i^2\beta_{i,j}J_0\left(\kappa_i\left\|\mathbf{r}_j\right\|\right) \tag{12}$$

where no intrusive measurement in the interior of the domain Ω was used.

## 3. Numerical Examples

Let us now apply the methodology presented in the previous section to the following dimensionless problem, where the geometry was given in Fig. (1)

$$\nabla^2 T = f(x_1,x_2) \quad \text{in } 0<x_1<1;\ 0<x_2<1 \tag{13.a}$$

$$T = T_0 \quad \text{at } x_1=0 \text{ and } x_1=1;\ 0<x_2<1 \tag{13.b}$$

$$T = T_0 \quad \text{at } x_2=0 \text{ and } x_2=1;\ 0<x_1<1 \tag{13.c}$$

where $T_0$ was taken as 0.0 and the measured heat fluxes are represented as

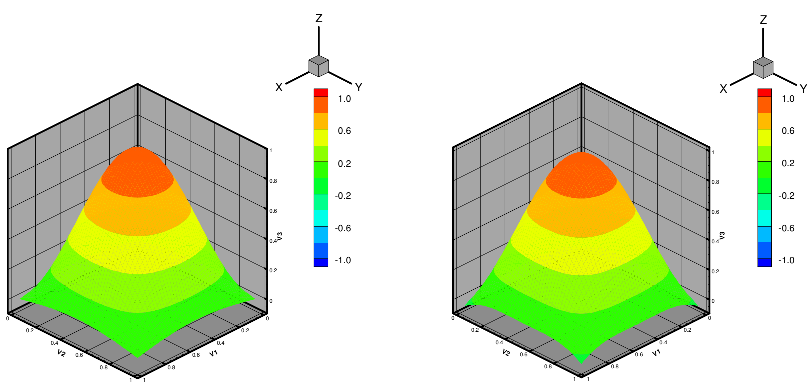
$$-K\frac{\partial T}{\partial \mathbf{n}} = q_{meas}(x_1,x_2) \quad \text{at } x_1=0 \text{ and } x_1=1;\ 0<x_2<1 \tag{14.d}$$

$$-K\frac{\partial T}{\partial \mathbf{n}} = q_{meas}(x_1,x_2) \quad \text{at } x_2=0 \text{ and } x_2=1;\ 0<x_1<1 \tag{14.e}$$

In order to test the above methodology, we used simulated measurements obtained by the solution of the direct problem, given by Eqs. (13.a,b,c), using a known distribution of source term $f(x_1,x_2)$. However, in order to avoid the so-called *inverse crime*, the methodology used to solve this direct problem was completely different from the methodology used to solve the inverse problem. In fact, we used the finite difference method to solve the direct problem and the MFS to solve the inverse problem. Thus, the measured heat fluxes at the boundaries already have some sort of error, due to the discretization used in the direct problem.

An open question in this MFS inverse solution gives respect to what number of centers ($x_j$) and frequencies $\kappa_i$ to use in Eqs. (7), (8) and (11). However, since the solution of Eq. (10) is quite fast (less than one second), we performed an automatic search of the number of centers ($x_j$) and frequencies $\kappa_i$ in all cases presented here, where 2<N<11 and $2^2$<M<$11^2$. The best values of the frequencies and centers were such that the residual of the linear system given by Eq. (10) was minimized. In all cases, we used 50 collocation points in each one of the four boundaries. After some tests, the estimation was insensitive to an increase in the number of the collocation points beyond this value.

For the first case, we considered a source term varying as a sine function in the $x_1$ and $x_2$ coordinates, and took an error free (except for the inherent error of the method used in the direct problem for the simulated measurements) measurement of the heat fluxes at the boundaries. Figure (2) shows the exact and estimated $f(x_1,x_2)$, where one can see a very good estimation, considering that we took only measurements at the boundaries.

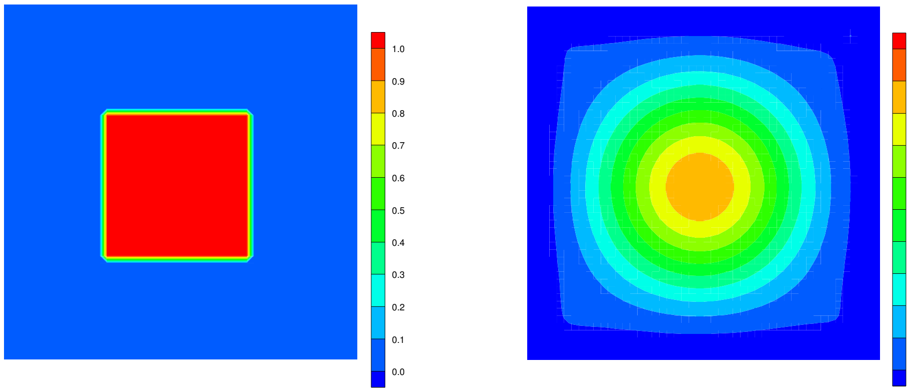



(a) (b)

Figure 2. Exact (a) and estimated (b) periodical source term

Once the method worked out for a smooth function as showed in Fig. (2), we decided to test discontinuous function, which are very difficult to capture. For this second test case we considered a square function with peak value equal to 1.0 and located exactly at the center of the medium, having length and weight equal to 0.5. Figure (3) shows the exact and estimated values of this source term, where one can see that the estimation is not able to capture the sharpness of the function, but captures reasonable well its peak value. It is also worth to note that the estimated location of the function is very well captured. Note that other techniques, as the Conjugate Gradient Method with Adjoint Problem (Colaço et al, 2003) was not able to estimate even smooth functions without intrusive measurements. For this estimation, we used 9 frequencies $\kappa_i$ and 16 centers $x_j$ inside the domain.

In order to facilitate the visualization, in all graphics of this paper the color scale was adapted to be the same for the exact and estimated functions. This leads to a better way to visualize the estimation than using 3D plots.




(a) (b)

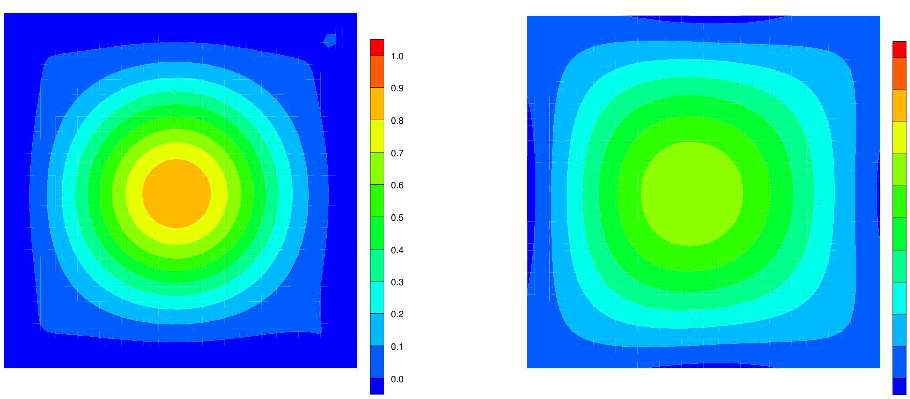
Figure 3. Exact (a) and estimated (b) symmetric square function, without measurement errors

Now, we used the same function of Fig. (3.a), but included errors in the measurements, which were obtained by using the following equation

$$\left(q_{meas}\right)_{err} = q_{meas} + \sigma\,\varepsilon \tag{15}$$

where $\sigma$ is the standard deviation of the measurements and $\varepsilon$ is a random number with uniform distribution between 0 and 1. Equation (15) was applied to each one of the $P$ measurement points.

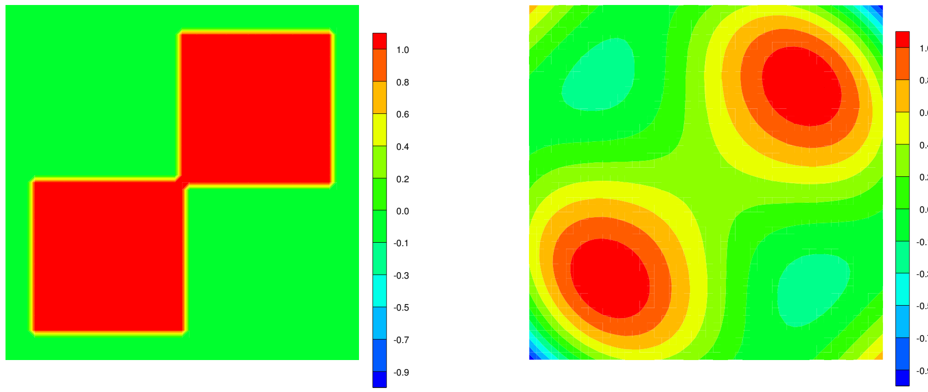
Figure (4) show the results for the estimation of the source term $g$ by using measurements with errors, whose standard deviation $\sigma$ was taken as 1% and 10% of the original measurements, for figures (4.a) and (4.b), respectively. Note that the estimation deteriorates when $\sigma$ increases. However, even for a large value of standard deviation, the location of the unknown source term is still captured. For these results we used 9 and 7 frequencies as well as 16 and 25 centers, respectively.




(a) (b)

Figure 4. Estimation of the symmetric square function with (a) $\sigma$=0.01 $q_{meas}$ and (b) $\sigma$=0.1 $q_{meas}$

The next test case involves the estimation of a source term which hasn't symmetry with respect to $x$ or $y$ axes. Also, in this test case, we tried to capture two discontinuities. The function is defined as a square function with peak value equal to one in the intervals showed in the Fig. (5). One can see that, although the shape of the function is not exactly estimated, the peak value of the source term and its locations are very well captured. The number of frequencies and centers used were 5 and 25, respectively.

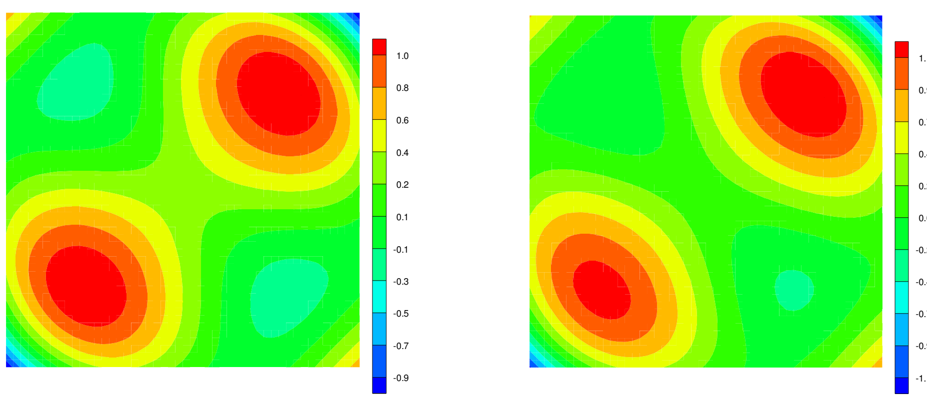



(a) (b)

Figure 5. Exact (a) and estimated (b) unsymmetric square function, without measurement errors

Figure (6) show the results for the estimation of the source term $g$ by using measurements with errors, whose standard deviation $\sigma$ was taken as 10% and 100% for the original measurements, for figures (6.a) and (6.b), respectively. Note that the estimation deteriorates when $\sigma$ increases (observe the values of the color scales). Note also that the estimation is worst in this case than in the case using symmetric source term, showed in Fig. (4). However, since the source terms are close to the boundaries (where the measurements were taken) we can obtain results even for large values of errors.

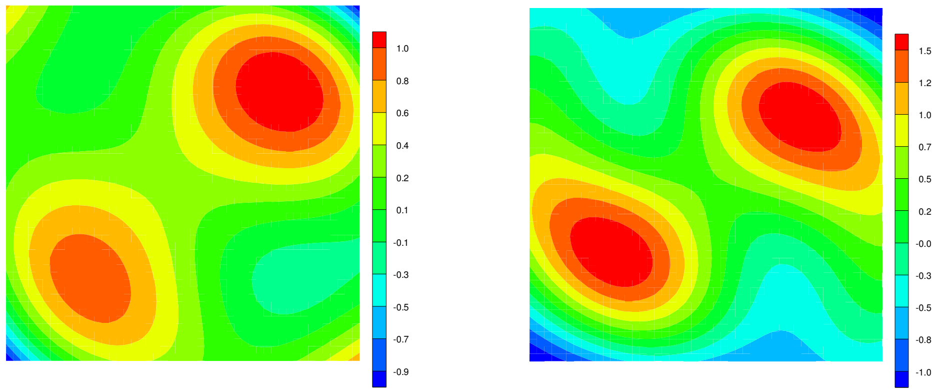
It is worth to note the large oscillations in the estimated function, having negative value at some locations. However, if one knows that the source term can be only positive for some specific problem; such plots give a good indication of the location of the peak values of the function. In fact, this information can also be used in the estimation in order to improve the results, although this was not done in this paper. For these results we used 5 frequencies and 36 centers.




(a) (b)

Figure 6. Estimation of the unsymmetric square function with (a) $\sigma$=0.1 and (b) $\sigma$=1.0

Finally, for the last test case, we tried to estimate the same function presented at Fig. (5.a), but with an underspecified boundary. In other words, the imposed boundary condition given by Eq. (1.b) and the measured heat flux given by Eq. (1.c) were unknown at some locations. This is a very extreme test case and Fig. (7.a) shows the result for a test case where nothing was known about the boundary $x$=0, while Fig. (7.b) shows the result where nothing was known for the boundaries $x$=0 and $x$=1, except the shape of the boundaries for both cases. Note, from Fig. (7.a) that the estimation only deteriorates close to $x$=0 (the underspecified boundary) and remains the same as showed in Fig. (5.b) close to $x$=1. However, the location of the peak values of the function is still well captured. Figure (7.b) shows a curious pattern, where the peak values are close to their exact location, but the function is stretched out in the $x$ direction (where both boundary conditions are underspecified). When one more boundary is underspecified, the results are too worst and are not shown here. For these results we used 5 and 9 frequencies as well as 36 and 9 centers, respectively.




(a) (b)

Figure 7. Underspecified boundary conditions at $x$=0 (a) and $x$=0, $x$=1 (b)

## 4. Conclusions

In this work we used a version of the Method of Fundamental Solutions, called the Boundary Knot Method to estimate an unknown source term in a linear heat conduction problem, by using only non-intrusive measurements.

Since we used simulated measurements, obtained from the solution of the direct problem for a known source term, two different methodologies were employed for the inverse and direct problems, in order to avoid the so-called inverse crime.

The MFS showed to be very fast and powerful to recover the location and peak values of discontinuous functions, even when more than one discontinuity was considered. It was also capable to recover the location of functions with measurement errors in the heat fluxes at the boundaries.

We also tested cases where some information of the boundaries was lost, except the location of the boundaries. Even when we did not know any boundary condition or measurement for parts of the boundary, the method was able to obtain a reasonable good estimation of the peak value and location of the unknown function.

## 5. Acknowledgements

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